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Perturbations of the Nonlinear Renewal Equation

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TO THE MEMORY OF NORMAN LEVINSON

1. The linear Volterra equation of convolution type

$$x(t) = f(t) + \int_0^t x(t-v)\, a(v)\, dv,$$

known as the renewal equation, has long been of interest, both because of its applications to growth and branching processes and because of the elegance of its mathematical theory. The basic facts about the asymptotic behavior of its solutions may be found in such sources as [1, 2, 10, 12,19]. The classical theorem of Paley and Wiener concerning its resolvent kernel [30, p. 58] has made it possible to study the asymptotic behavior of solutions of nonlinear perturbations of the renewal equation as well, especially their relation to solutions of the unperturbed equation; see, for example, [25–28].

The nonlinear renewal equation

$$x(t) = f(t) + \int_0^t g(x(t-v))\, a(v)\, dv \tag{1}$$

has arisen in heat transfer problems [24, 29, 31], superfluidity [20], nuclear reactor dynamics [16–18], and population growth problems [5, 8], as well as in nonlinear branching processes [6, 7]. Because of its importance in diverse applications, the nonlinear renewal equation has been studied extensively in recent years, and as a result of the theory developed in the papers cited above together with such other works as [11, 13–15, 19, 21–23], much is known about the asymptotic behavior of its solutions. These results involve methods quite different from the linearization techniques used to study perturbations of the linear renewal equation.

The purpose of this paper is to examine the effects of perturbations on the solutions of the nonlinear renewal equation, particularly on the limits of solutions as $t \to \infty$. We shall see that these limits may be separated into two classes with very different behavior under perturbations in much the same way that critical points of autonomous systems of ordinary differential equations may be classified as asymptotically stable or unstable. However, the difference between the two classes arises differently from the ordinary differential equation case, in the analog of perturbations of the equation rather than of perturbations of the initial conditions.

The applications of the nonlinear renewal equation to population and branching problems are for integrable kernels, $\int_0^\infty a(v)\, dv < \infty$, while the applications to physical problems usually arise from transformations of the heat equation and have kernel $a(v) = Kv^{-1/2}$. We shall concentrate on the case of integrable kernels, but in Section 6 we shall also examine the rather different situation when $\int_0^\infty a(v)\, dv = \infty$. We shall also make an attempt to unify the two classes of problems, but here our results are purely formal.

2. In the study of the nonlinear renewal equation it has turned out to be advantageous to treat separately the questions of whether all solutions are bounded and of whether all bounded solutions tend to limits as $t \to \infty$. There are many different boundedness theorems, with hypotheses appropriate to the diverse applications for which they are intended; see, for example, [4, 15, 22, 24, 29]. In the first part of this paper we will concentrate on the limits of bounded solutions, taking for granted that some sufficient conditions for boundedness are satisfied. For example, under the conditions H_f, H_a, H_g below it is sufficient to require $xg(x) \leqslant 0$ for $-\infty < x < \infty$ [15], or $G(x) = -\int_0^x g(\xi)\, d\xi \to \infty$ as $|x| \to \infty$ and $|g(x)| \leqslant K(1 + G(x))$ for $-\infty < x < \infty$ [21].

In studying the limits as $t \to \infty$ of solutions of (1), we will make the following hypotheses on the functions f, a, and g throughout the first five sections of the paper. In Section 6 we will consider a somewhat different situation, with different hypotheses.

H_f: We assume that $f(t)$ is continuous on $0 \leqslant t < \infty$, and that

$$f(\infty) = \lim_{t\to\infty} f(t)$$

exists.

H_a: We assume that $a(v)$ is continuous, nonnegative, and monotone nonincreasing on $0 \leqslant v < \infty$, and that

$$\int_0^\infty a(v)\,dv < \infty.$$

H_g: We assume that $g(x)$ is continuously differentiable on $-\infty < x < \infty$ and that there is no interval on which $g'(x) \int_0^\infty a(v)\,dv \equiv 1$.

When we speak of all solutions of (1), we shall always mean the set of solutions of all equations of the form (1) with fixed a and g and any f satisfying H_f.

In Section 3 we shall discuss a nonlinear renewal equation of the form

$$x(t, \lambda) = f(t, \lambda) + \int_0^t g(x(t-v, \lambda))\, a(v)\,dv \tag{2}$$

in which the forcing function depends on a parameter λ. For this equation, we replace H_f by the hypotheses

H_f': We assume that $f(t, \lambda)$ is continuous and of bounded variation in t on $0 \leqslant t < \infty$ for λ in some interval $0 \leqslant \lambda < \lambda_0$, which implies the existence of

$$f(\infty, \lambda) = \lim_{t\to\infty} f(t, \lambda)$$

for $0 \leqslant \lambda < \lambda_0$. Further, we assume that $(d/d\lambda) f(\infty, \lambda)$ exists for $0 \leqslant \lambda < \lambda_0$.

It is easy to see that under these conditions if $x(t)$ is a solution of (1) such that $x_\infty = \lim_{t\to\infty} x(t)$ exists, then

$$\lim_{t\to\infty} \int_0^t g(x(t-v))\, a(v)\,dv = g(x_\infty) \int_0^\infty a(v)\,dv.$$

It follows that x_∞ must be a solution of the equation

$$x_\infty = f(\infty) + g(x_\infty) \int_0^\infty a(v)\,dv. \tag{3}$$

Similarly, if $x(t, \lambda)$ is a solution of (2) such that $x_\infty(\lambda) = \lim_{t\to\infty} x(t, \lambda)$ exists, then $x_\infty(\lambda)$ must be a solution of the equation

$$x_\infty = f(\infty, \lambda) + g(x_\infty(\lambda)) \int_0^\infty a(v)\,dv. \tag{4}$$

The following result has been established for limits of solutions of (1). The obvious analog holds for limits of solutions of (2) for each fixed λ.

THEOREM 1 [23]. *Suppose the conditions* H_f, H_a, and H_g *are satisfied. Then every bounded solution* $x(t)$ *of* (1) *satisfies*

$$\lim_{t\to\infty}\left[x(t) - g(x(t))\int_0^\infty a(v)\,dv\right] = f(\infty). \tag{5}$$

The condition that there be no interval on which $g'(x)\int_0^\infty a(v)\,dv = 1$ implies that the roots of (3) are isolated. It then follows easily from (5) that $x(t)$ tends to a limit as $t \to \infty$. This yields the following result.

COROLLARY. *Suppose the conditions* H_f, H_a, *and* H_g *are satisfied. Then every bounded solution* $x(t)$ *of* (1) *tends as* $t \to \infty$ *to a limit* x_∞ *which is a root of Eq.* (3).

3. For the nonlinear renewal equation, Eq. (2), depending on a parameter λ, we are interested in the dependence of the limit $x_\infty(\lambda) = \lim_{t\to\infty} x(t, \lambda)$ on λ. By considering Eq. (4) and applying the implicit function theorem, we obtain the following result, essentially contained in [5].

THEOREM 2. *Suppose that the hypotheses* H_f', H_a, *and* H_g *are satisfied. Let* $x(t, 0)$ *be a solution of* (2) *for* $\lambda = 0$ *which tends to a limit* $x_\infty(0)$ *as* $t \to \infty$ *and suppose*

$$g'(x_\infty(0))\int_0^\infty a(v)\,dv \neq 1.$$

Then there is an interval $0 \leqslant \lambda < \lambda_0$ *on which* $x_\infty(\lambda)$ *is a differentiable function of* λ, *and*

$$\frac{d}{d\lambda}x_\infty(\lambda) = \frac{(df/d\lambda)(\infty, \lambda)}{1 - g'(x_\infty(\lambda))\int_0^\infty a(v)\,dv}. \tag{6}$$

The value λ *is determined as the smaller of* λ_0 *in hypothesis* H_f' *and the least positive solution* λ_1 *of* $g'(x_\infty(\lambda_1))\int_0^\infty a(v)\,dv = 1$.

Geometrically, we find $x_\infty(\lambda)$ by finding the abscissas of the intersections of the curve $y = g(x)$ and the straight line $y = [x - f(\infty, \lambda)]/\int_0^\infty a(v)\,dv$, taking the intersection whose abscissa is

$x_\infty(\lambda)$ for $\lambda = 0$, and varying continuously as λ is increased. If $(df/d\lambda)(\infty, \lambda) > 0$, then increasing λ means moving the straight line downward, which results in an increase in $x_\infty(\lambda)$ if $g'(x_\infty(\lambda)) \int_0^\infty a(v)\, dv < 1$ but a decrease in $x_\infty(\lambda)$ if $g'(x_\infty(\lambda)) \int_0^\infty a(v) dv > 1$. Thus if $g'(x_\infty(\lambda)) \times \int_0^\infty a(v)\, dv > 1$, an increase in the forcing function $f(t, \lambda)$ leads to a decrease in the limit $x_\infty(\lambda)$ of the solution of (2). Such behavior is somewhat surprising, and we shall see in Section 5 that in this case a small perturbation of the nonlinear renewal equation may have a large effect on the solution. This suggests that in applications it may be appropriate to give physical validity only to equilibrium values x_∞ with $g'(x_\infty) \int_0^\infty a(v)\, dv < 1$, even though equilibrium values for which $g'(x_\infty) \int_0^\infty a(v)\, dv > 1$ are possible.

If there is a value λ_1 for which $g'(x_\infty(\lambda_1)) \int_0^\infty a(v)\, dv = 1$, then for $\lambda = \lambda_1$ the curve $y = g(x)$ and the line $y = [x - f(\infty, \lambda)]/\int_0^\infty a(v)\, dv$ are tangent. A further increase in λ causes the disappearance of the intersection, unless the curve $y = g(x)$ has an inflection point at the point of tangency, with the result that there must be a discontinuity in $x_\infty(\lambda)$ at λ_1. Such an occurrence is called a catastrophe because of the physical connotation.

The special case of Theorem 2 in which $f(\infty, \lambda)$ is independent of λ will be needed in Section 4. For this reason, we formulate it explicitly.

COROLLARY 1. *Suppose that the hypotheses H_f', H_a, and H_g are satisfied and that $f(\infty, \lambda)$ is independent of λ. Let $x(t, 0)$ be a solution of* (2) *for $\lambda = 0$ which tends to a limit x_∞ as $t \to \infty$, and suppose*

$$g'(x_\infty) \int_0^\infty a(v)\, dv \neq 1.$$

Then $\lim_{t\to\infty} x(t, \lambda) = x_\infty$ *for* $0 \leqslant \lambda < \lambda_0$.

To prove this corollary, we merely observe that $(d/d\lambda)\, x_\infty(\lambda) = 0$ by (6), and therefore the limit x_∞ is constant.

Another useful consequence of Theorem 2 is the analog for the nonlinear renewal equation of a stability theorem for ordinary differential equations. The ordinary differential equation $x' = g(t, x)$, $x(0) = x_0$ is equivalent to the Volterra integral equation

$$x(t) = x_0 + \int_{t_0}^t g(s, x(s))\, ds. \tag{7}$$

Stability of a solution of the differential equation is the property that a small change in x_0 has only a small effect on the solution. The analogous property for the nonlinear renewal equation (1) is that a small change in $f(t)$ has only a small effect on the solution.

COROLLARY 2. *Suppose that the hypotheses H_f, H_a, and H_g are satisfied. Let $x(t)$ be a solution of* (1) *tending to a limit x_∞ for which*

$$g'(x_\infty) \int_0^\infty a(v)\, dv \neq 1.$$

Let $p(t)$ be continuous and of bounded variation on $0 \leqslant t < \infty$, and let $y(t)$ be a bounded solution of the perturbed equation,

$$y(t) = f(t) + p(t) + \int_0 g(y(t-v))\, a(v)\, dv. \tag{8}$$

Then there exist $\delta > 0$, $K > 0$ such that if $|p(t)| < \delta$ for $0 \leqslant t < \infty$, then $|y(t) - x(t)| < K\delta$ for all sufficiently large t. If $p(\infty) = \lim_{t\to\infty} p(t) = 0$, then $\lim_{t\to\infty} y(t) = x_\infty$.

Proof. We consider Eq. (2) with $f(t, \lambda) = f(t) + \lambda p(t)$, so that Eq. (8) is (2) with $\lambda = 1$ and $f(\infty, \lambda) = p(\infty)$. Since $x_\infty(\lambda)$ is continous in λ, we can choose $\delta > 0$ small enough that $g'(x_\infty(\lambda)) \int_0^\infty a(v)\, dv \neq 1$ for $0 \leqslant \lambda < 1$, and therefore (6) holds for $0 \leqslant \lambda \leqslant 1$. By Theorem 2, $y(t)$ tends to a limit y_∞ as $t \to \infty$, and, in view of (6),

$$y_\infty - x_\infty = \int_0^1 \frac{dx_\infty(\lambda)}{d\lambda}\, d\lambda = p(\infty) \int_0^1 \frac{d\lambda}{1 - g'(x_\infty(\lambda)) \int_0^\infty a(v)\, dv}. \tag{9}$$

Since $g'(x_\infty(\lambda)) \int_0^\infty a(v)\, dv \neq 1$, the integral on the right side of (9) is bounded. Thus there exists a constant K such that $|y_\infty - x_\infty| < K|p(\infty)| < K\delta$ and this implies that $|y(t) - x(t)| < K\delta$ for some constant K and all large t. If $p(\infty) = 0$, it follows directly from (9) that $y_\infty = x_\infty$.

4. A perturbation theorem for an ordinary differential equation in the equivalent form (7) is a statement that a change in $g(t, x)$ which is small in some sense has only a small effect on the solution. We shall obtain an analogous result for the nonlinear renewal equation, namely, a theorem for a perturbed equation

$$y(t) = f(t) + \int_0^t g(y(u))\, a(t-u)\, du + \int_0^t \psi(t, u, y(u))\, du$$

with $\psi(t, u, y)$ small in a suitable sense. For this we require a nonlinear variation of constants formula for Volterra equations, as obtained in [3]. We give a brief resume of this formula in order to analyze the properties needed for its application here.

We wish to compare the solution $y(t)$ of the perturbed integral equation

$$y(t) = f(t) + \int_{t_0}^{t} g(t, s, y(s))\, ds + \int_{t_0}^{t} \psi(t, s, y(s))\, ds \tag{10}$$

with the solution $x(t)$ of the unperturbed unperturbed equation

$$x(t) = f(t) + \int_{t_0}^{t} g(t, s, x(s))\, ds. \tag{11}$$

We begin by considering the equivalent integro-differential equations

$$y'(t) = f'(t) + g(t, t, y(t)) + \int_{t_0}^{t} g_t(t, s, y(s))\, ds + \int_{t_0}^{t} \psi_t(t, s, y(s))\, ds$$
$$+ \psi(t, t, y(t)), \qquad y(t_0) = f(t_0)$$

and

$$x'(t) = f'(t) + g(t, t, x(t)) + \int_{t_0}^{t} g_t(t, s, x(s))\, ds, \qquad x(t_0) = f(t_0), \tag{12}$$

respectively. We denote the respective solutions by $y(t, t_0, y(t_0))$ and $x(t, t_0, x(t_0, x(t_0))$ to indicate their dependence on initial values. Let $U(t, t_0, x(t_0))$ be the solution of the variational equation

$$\begin{aligned}\frac{\partial}{\partial t} & U(t, t_0, x(t_0)) \\ &= g_x(t, t, x(t))\, U(t, t_0, x(t_0)) \\ &\quad + \int_{t_0}^{t} g_{tx}(t, s, x(s))\, U(s, t_0, x(t_0))\, ds, \qquad U(t_0, t_0, x(t_0)) = 1\end{aligned}$$

of (12) with respect to the solution $x(t)$. Then after a computation of $(d/ds)\, x(t, s, y(s))$ and an integration, we obtain the formula

$$\begin{aligned} y(t, t_0, & f(t_0)) - x(t, t_0, f(t_0)) \\ &= \int_{t_0}^{t} U(t, s, y(s)) \left(\psi(s, s, y(s)) + \int_{t_0}^{s} \psi_t(s, u, y(u))\, du\right) ds \\ &= \int_{t_0}^{t} U(t, s, y(s)) \frac{d}{ds} \left(\int_{t_0}^{s} \psi(s, u, y(u))\, du\right) ds.\end{aligned}$$

Here, $U(t, s, y(s))$ denotes the solution of the variational equation of the integro-differential equation

$$x'(t) = f'(t) + g(t, t, x(t)) + \int_s^t g_t(t, u, x(u))\, du, \qquad x(s) = f(s)$$

or of the equivalent integral equation

$$x(t) = f(t) + \int_s^t g(t, u, x(u))\, du, \qquad t \geqslant t_0 . \tag{13}$$

This integral equation also may be written

$$x(t) = f_s(t) + \int_{t_0}^t g(t, u, x(u))\, du, \qquad t \geqslant t_0$$

where

$$f_s(t) = f(t) - \int_{t_0}^s g(t, u, x(u))\, du. \tag{14}$$

Let $x_s(t)$ be the solution of (13); then $U(t, s, y(s))$ is the solution of the variational equation of (13) with respect to its solution $x_s(t)$. To indicate the dependence on the class of equations (13), we write $U(t, s, x_s)$ rather than $U(t, s, y(s))$ when studying integral equations. Then $U(t, s, x_s)$ satisfies the variational equation

$$U(t, s, x_s) = 1 + \int_s^t g_x(t, u, x_s(u))\, U(u, s, x_s)\, du,$$

and the solutions of (10) and (11) are related by

$$\begin{aligned} y(t) - x(t) &= \int_{t_0}^t U(t, s, x_s) \left\{ \psi(s, s, y(s)) + \int_{t_0}^s \psi_t(s, u, y(u))\, du \right\} ds \\ &= \int_{t_0}^t U(t, s, x_s) \frac{d}{ds} \left\{ \int_{t_0}^s \psi(s, u, y(u))\, du \right\} ds, \end{aligned} \tag{15}$$

for $t \geqslant t_0$. This is the desired variation of constants formula.

In the case where the unperturbed equation (10) is the nonlinear renewal equation, so that $g(t, u, x) = g(x)\, a(t - u)$, if the hypotheses H_f, H_a, and H_g are satisfied, we see that $f_s(t)$ in (14) is continuous and of bounded variation, Since H_a implies $\lim_{t\to\infty} a(t - u) = 0$ for $0 \leqslant u \leqslant s$, we see also that $f_s(\infty) = f(\infty)$. It now follows from

Theorem 2, Corollary 1 that if $x(t)$ is a solution of (1) tending to the limit x_∞ and if H_f, H_a, and H_g are satisfied, then the solution $x_s(t)$ of

$$x_s(t) = f_s(t) + \int_0^t g(x(u))\, a(t-u)\, du \tag{16}$$

tends to x_∞ as $t \to \infty$ for each s, provided $g'(x_\infty) \int_0^\infty a(v)\, dv \neq 1$. This fact is useful in the analysis of the behavior of the solutions of the variational equation,

$$U(t, s, x_s) = 1 + \int_s^t g'(x_s(u))\, a(t-u)\, U(u, s, x_s)\, du, \tag{17}$$

of (16) with respect to the solution $x_s(t)$.

THEOREM 3. *Suppose that H_f, H_a, and H_g are satisfied. Let $x(t)$ be a bounded solution of* (1) *which tends to the limit x_∞ as $t \to \infty$. Then the solution $U(t, s, x_s)$ of the variational equation* (17) *of* (16) *with respect to the solution $x_s(t)$ tends to a limit*

$$U_\infty = \frac{1}{1 - g'(x_\infty) \int_0^\infty a(v)\, dv}$$

independent of s as $t \to \infty$ if

$$g'(x_\infty) \int_0 a(v)\, dv < 1 \tag{18}$$

and is unbounded if

$$g'(x_\infty) \int_0^\infty a(v)\, dv > 1. \tag{19}$$

Proof. We suppose that $g'(x_\infty) \neq 0$; the case $g'(x_\infty) = 0$ will be treated by a slightly different argument. We have remarked that in both the cases (18) and (19), $\lim_{t\to\infty} x_s(t) = x_\infty$. Since $g'(x)$ is continuous, for any $\epsilon > 0$ we may choose $\tau \geqslant 0$ large enough that

$$| g'(x_s(u)) - g'(x_\infty)| < \epsilon\, | g'(x_\infty)|, \qquad u \geqslant \tau$$

Then we may write (17) in the form

$$U(t, s, x_s) = f_\tau(t, s) + \int_\tau^t g'(x_s(u))\, a(t-u)\, U(u, s, x_s)\, du,$$

where

$$f_\tau(t, s) = 1 + \int_s^\tau g'(x_s(u))\, a(t-u)\, U(u, s, x_s)\, du.$$

Since $a(t-u)$ is decreasing and integrable, we see that $f_\tau(t, s)$ is continuous and of bounded variation as a function of t and tends to the limit 1 as $t \to \infty$ for any fixed τ, s. We now rewrite (17) as

$$U(t, s, x_s) = f_\tau(t, s) + \int_\tau^t g'(x_\infty)\, a(t-u)\, U(u, s, x_s)\, du$$
$$+ \int_\tau^t g'(x_\infty)\, a(t-u)\, \beta(u)\, U(u, s, x_s)\, du \tag{20}$$

with

$$|\beta(u)| = \left| \frac{g'(x_s(u)) - g'(x_\infty)}{g'(x_\infty)} \right| < \epsilon, \qquad u > \tau.$$

We consider (20) as a perturbation of the linear renewal equation

$$z_s(t) = f(t, s) + \int_\tau^t g'(x_\infty)\, a(t-u)\, z_s(u)\, du. \tag{21}$$

By the variation of constants formula for linear integral equations [24, 26] we may write

$$U(t, s, x_s) - z_s(t) = -\int_\tau^t b(t-u)\, \beta(u)\, U(u, s, x_s)\, du, \tag{22}$$

where $b(t)$ is the resolvent kernel corresponding to the kernel $g'(x_\infty)\, a(t)$. By the Paley–Wiener theorem [29, p. 58]; see [26] or [28]), $b(t)$ is integrable on $[0, \infty]$ if (18) is satisfied, and this together with the representation of the solution of (21) in terms of the resolvent kernel [25, pp. 189–191] implies that $z_s(t)$ is bounded for $\tau \leqslant t < \infty$, for each s. We now fix $\epsilon > 0$ so that $\epsilon \int_0^\infty b(v)\, dv = \theta < 1$; then (22) gives

$$|U(t, s, x_s)| = \sup_{\tau \leqslant t < \infty} |z_s(t)| + \sup_{\tau \leqslant u \leqslant t} |U(u, s, x_s)|,$$

from which we obtain

$$\sup_{\tau \leqslant u < t} |U(u, s, x_s)| \leqslant \sup_{\tau \leqslant t < \infty} |z_s(t)|/(1-\theta), \qquad t \leqslant \tau.$$

This implies that $U(t, s, x_s)$ is bounded on $\tau \leqslant t < \infty$. By Theorem 2, or directly from the representation for $z_s(t)$ in terms of the resolvent kernel, we see that

$$\lim_{t\to\infty} z_s(t) = \frac{1}{1 - g'(x_\infty)\int_0^\infty a(v)\,dv}.$$

Since

$$| U(t, s, x_s) - z_s(t)| \leqslant \sup_{\tau\leqslant u<\infty} | U(u, s, x_s)| \int_\tau^t | b(t-u)| \cdot | \beta(u)|\,du$$

from (22), and since it follows from $\lim_{u\to\infty} \beta(u) = 0$ and $b(t)\in L^1(0, \infty)$ that $\lim_{t\to\infty} \int_\tau^t | b(t-u)| \cdot | \beta(u)|\,du = 0$, we see that

$$\lim_{t\to\infty} U(t, s, x_s) = \lim_{t\to\infty} z_s(t) = U_\infty .$$

The limit here exists for any fixed s. The case $g'(x_\infty) = 0$ which we have excluded is trivial since in this case the kernel of (21) is zero. If (19) is satisfied, the Paley–Wiener theorem implies that $z_s(t)$ is unbounded; in fact, $z_s(t)$ grows exponentially. If $U(t, s, x_s)$ were bounded, the integral on the right side of (22) would be bounded as well, and this would imply that $z_s(t)$ must be bounded. This contradiction shows that $U(t, s, x_s)$ must be unbounded in the case (19), and completes the proof of Theorem 3.

5. Theorem 3 suggests that there is a significant difference between cases (18) and (19). In the special case $f(t) = x_0 e^{-\alpha t}$, $a(v) = e^{-\alpha v}$, Eq. (1) is equivalent to the ordinary differential equation $x' = g(x) - \alpha x$, $x(0) = x_0$. Let x_∞ be a critical point of this equation, so that $g(x_\infty) = \alpha x_\infty$. Then (18), or $g'(x_\infty) < \alpha$, implies the asymptotic stability of the critical point x_∞, while (19), or $g'(x_\infty) > \alpha$, implies its instability. It is not true in general for (1) that a nonconstant solution cannot tend to a limit x_∞ for which (19) is satisfied, as may be seen by the linear example $a(v) = e^{-\alpha v}$, $g(x) = x$, $f(t) = e^{-\alpha t}(\cos t - \sin t)$, with $x(t) = e^{-\alpha t}\cos t \to 0$ for all $\alpha > 0$. However, there is a significant difference between cases (18) and (19) in the effect of small perturbations on the solution.

For an ordinary differential equation written in the integrated form (7), such properties as total stability and integral stability are statements about the effect of a change in $g(t, x)$ which is small in some sense. For example, total stability ir the property that addition to $g(t, x)$ of a term $h(t, x)$ such that for every $\epsilon > 0$ there exists $\delta > 0$ with $| h(t, x)| < \delta$ for $t \geqslant 0$, $| x | < \epsilon$ has only a small effect on the solution. An analogous property holds for the nonlinear renewal equation if (18) is satisfied, but not if (19) is satisfied.

THEOREM 4. *Suppose that H_f, H_a, and H_g are satisfied. Let $x(t)$ be a bounded solution of* (1) *which tends to the limit x_∞ as $t \to \infty$. Let $y(t)$ be a solution of*

$$y(t) = f(t) + \int_0^t g(y(t-v))\, a(v)\, dv + \int_0^t \psi(t, u, y(u))\, du.$$

If (18) *is satisfied, then there exists $\delta > 0$ such that for any ψ with $\int_0^t |\psi(t, u, y(u))|\, du < \delta$ whenever $|y(u) - x(u)|$ is sufficiently small for $0 \leqslant u \leqslant t$, $0 \leqslant t < \infty$, we have $|y(t) - x(t)| < K\delta$ for some constant $K > 0$ and all large t. If $\lim_{t\to\infty} \int_0^t \psi(t, u, y(u))\, du = 0$ for all such functions y, then $\lim_{t\to\infty} y(t) = x_\infty$. If* (19) *is satisfied, there are arbitrarily small perturbations for which $y(t)$ does not tend to x_∞.*

Proof. In case (18), Theorem 3 gives the existence of $\lim_{t\to\infty} U(t, s, x_s) = U_\infty$. The nonlinear variation of constants formula (15) gives

$$y(t) - x(t) = \int_0^t U(t, s, x_s) \frac{d}{ds} \left\{ \int_0^s \psi(s, u, y(u))\, du \right\} ds,$$

which is approximated for large t by

$$U_\infty \int_0^t \frac{d}{ds} \left\{ \int_0^s \psi(s, u, y(u))\, du \right\} ds = U_\infty \int_0^t \psi(t, u, y(u))\, du,$$

and the result follows. We must choose δ small enough to avoid the possibility of a limit y_∞ of $y(t)$ for which $g'(y_\infty) \int^\infty a(v)\, dv \geqslant 1$. In case (19) we may choose $\psi(t, s, y)$ a function of s only which has the same sign as $U(t, s, x_s)$ for large t, and then since $U(t, s, x_s)$ is unbounded we see that $y(t) - x(t) = \int_0^t U(t, s, x_s)\, \psi(s)\, ds$ cannot be small for large t.

In the special case $\psi(t, u, y) = h(y)\, a(t-u)$ with $|h(y)| < \delta$,

$$\begin{aligned} \int_0^t |\psi(t, u, y(u))|\, du &= \int_0^t |h(y(u))|\, a(t-u)\, du \\ &\leqslant \delta \int_0^t a(t-u)\, du \leqslant \delta \int_0^\infty a(v)\, dv. \end{aligned}$$

Thus Theorem 4 is applicable, and we obtain the following result.

COROLLARY. *Suppose that* H_f, H_a, *and* H_g *are satisfied. Let* $x(t)$ *be a bounded solution of* (1) *which tends to the limit* x_∞ *as* $t \to \infty$. *Let* $y(t)$ *be a solution of*

$$y(t) = f(t) + \int_0^t \{g(y(t-v)) + h(y(t-v))\}\, a(v)\, dv.$$

If (18) *is satisfied, then there exists* $\delta > 0$ *such that* $|h(y)| < \delta$ *for all* y *implies* $|y(t) - x(t)| < K\delta$ *for some constant* K *and all large* t.

This property is the analog for the nonlinear renewal equation of total stability for ordinary differential equations.

By means of Theorem 4, we can give another approach to the problem treated in Theorem 2, Corollary 2. To study the effect of a perturbation $p(t)$ as in Theorem 2, Corollary 2, we may write $p(t) = p(0) + \int_0^t p'(s)\, ds$. The effect of the constant $p(0)$ can be treated as in Theorem 2, Corollary 2, and is seen to be

$$p(0) \int_0^1 \frac{d\lambda}{1 - g'(x_\infty(\lambda)) \int_0^\infty a(v)\, dv},$$

which is approximately $U_\infty p(0)$ if $p(0)$ is small enough that $g'(x_\infty(\lambda))$ varies little for $0 \leqslant \lambda \leqslant 1$. The effect of the term $\int_0^t p'(s)\, ds$ can be measured by Theorem 4, and is seen to be $\int_0^t U(t, s, x_s)\, p'(s)\, ds$, which is approximately $U_\infty \int_0^\infty p'(s)\, ds = U_\infty[p(\infty) - p(0)]$. Thus, the total effect of the perturbation on $y_\infty - x_\infty$ is $U_\infty p(\infty)$, just as obtained in Theorem 2, Corollary 2. The advantage of breaking the perturbation up into two parts and using Theorem 4 is that it suggests a possible approach to more general nonlinear integral equation than the renewal equation. In order to study the effect of perturbations on an equation

$$x(t) = f(t) + \int_0^t g(t, s, x(s))\, ds, \tag{23}$$

it may be sufficient to obtain information about the solution of

$$x(t, \lambda) = f(t, \lambda) + \int_0^t g(t, s, x(s, \lambda))\, ds,$$

where $\lim_{t\to\infty} f(t, \lambda) = \lim_{t\to\infty} f(t)$ in order to be able to treat the variational equation of (23) and the solutions of

$$x(t) = f(t) + c + \int_0^t g(t, s, x(s))\, ds$$

where c is a constant. These two special cases may be much easier to study than more general perturbations, and the nonlinear variation of constants formula may supply the means to pass from these special cases to the more general situation.

6. The problems of heat conduction and nuclear reactor dynamics which led to much of the work on the nonlinear renewal equation [16–18, 24, 29, 31] were in situations where $\int_0^\infty a(v)\,dv = \infty$ and $f(t)$ is constant in (1). Accordingly, we consider some perturbation questions under the following hypotheses, somewhat different from the hypotheses H_f, H_a, H_g introduced in Section 2.

H_p': We assume that $p(t)$ is continuous and of bounded variation on $0 \leqslant t < \infty$.

H_a': We assume that $a(v)$ is continuous, nonnegative, and monotone nonincreasing on $0 < v < \infty$, integrable at 0, and that

$$\int_0^\infty a(v)\,dv = \infty.$$

H_g': We assume that $g(x)$ is continuously differentiable on $-\infty < x < \infty$ and that there is no interval on which $g(x) \equiv 0$.

We consider the uperturbed equation

$$x(t) = c + \int_0^t g(t - v))\,a(v)\,dv \tag{24}$$

and the perturbed equation

$$y(t) = c + p(t) + \int_0^t g(t - v))\,a(v)\,dv. \tag{25}$$

The following variant of Theorem 1 has been established in [22].

THEOREM 5. *Suppose the conditions H_p', H_a', and H_g' are satisfied. Then every bounded solution $y(t)$ of* (25) *satisfies*

$$\lim_{t\to\infty} g(y(t)) = 0.$$

The condition H_g' implies that the roots of $g(y) = 0$ are isolated. Just as for Theorem 1, it follows that $y(t)$ tends to a limit, and we obtain the following result.

COROLLARY. *If the conditions H_p', H_a', and H_g' are satisfied, then every bounded solution $y(t)$ of* (25) *tends as $t \to \infty$ to a limit x_∞ which is a root of the equation*

$$g(x_\infty) = 0. \tag{26}$$

By considering a perturbed forcing function $\lambda p(t)$ $(0 \leqslant \lambda \leqslant 1)$ and varying λ continuously we may use the fact that the roots of (26) are independent of the forcing function to argue exactly as in Theorem 2, Corollary 1. This yields the following result.

THEOREM 6. *Suppose the hypotheses H_p', H_a', and H_g' are satisfied. Let* (24) *have a bounded solution $x(t)$ which tends as $t \to \infty$ to a limit x_∞. Then for every $p(t)$ such that the solution $y(t)$ of* (25) *is bounded, we have* $\lim_{t\to\infty} y(t) = x_\infty$.

The content of Theorem 6 is essentially that a boundedness result for (25) for a given class of functions $p(t)$ yields a result on the limits of solutions of (25) for this class of perturbations of (24). The global boundedness result of [21] (Theorem 1) can be modified to give a local result, or we can give the following essentially equivalent local result.

THEOREM 7. *Suppose the hypotheses H_p', H_a', and H_g' are satisfied. Let* (24) *have a bounded solution $x(t)$ which tends as $t \to \infty$ to a limit x_∞ and suppose $g'(x_\infty) < 0$. Then there is an interval I with x_∞ in its interior such that if $c \in I$ and the total variation $\int_0^\infty |p'(t)|\,dt$ is sufficiently small, the solution $y(t)$ of* (25) *tends to x_∞ as $t \to \infty$.*

Proof. In view of Theorem 6, we need only prove boundedness of solutions $y(t)$ of (25). The conditions $g(x_\infty) = 0$ *and* $g'(x_\infty) < 0$ imply that there is an open interval I about x_∞ on which $g(y)$ is strictly decreasing. Using the mean value theorem, we see that on this interval I there exist constants $\alpha > 0$, $\beta > 0$ such that

$$\beta(y - x) > -g(y) > \alpha(y - x), \qquad y \in I, y > x_\infty$$
$$-\beta(y - x) > g(y) > -\alpha(y - x), \qquad y \in I, y < x_\infty .$$

From this we deduce that

$$G(y) = -\int_0^y g(\xi)\,d\xi \geqslant \frac{\alpha}{2}(y - x_\infty)^2, \qquad y \in I \tag{27}$$

and

$$\beta \mid y - x_\infty \mid > \mid g(y) \mid > \alpha \mid y - x_\infty \mid, \qquad y \in I. \tag{28}$$

Because of H_a', the quantity

$$E(y(t)) = G(y(t)) + \tfrac{1}{2} \int_0^t [g(y(u))]^2 a(t-u)\, du$$

is greater than $G(y(t))$. We calculate

$$\frac{d}{dt} E(y(t)) = -g(y(t))\, y'(t) + \frac{1}{2} a(0)[g(y(t))]^2 + \frac{1}{2} \int_0^t a'(t-u)[g(y(u))]^2\, du$$

and since

$$\begin{aligned} -g(y(t))\, y'(t) &= -g(t))\, p'(t) - a(0)[g(y(t))]^2 - g(y(t)) \int_0^t g(y(u))\, a'(t-u)\, du \\ &\quad - \tfrac{1}{2} \int_0^t [g(y(t))]^2 a'(t-u)\, du - \tfrac{1}{2} \int_0^t [g(y(u))]^2 a'(t-u)\, du \\ &= -g(y(t))\, p'(t) - \tfrac{1}{2} a(t)[g(y(t))]^2 - \tfrac{1}{2} a(0)[g(y(t))]^2 \\ &\quad + \tfrac{1}{2} \int_0^t a'(t-u)[g(y(t)) - g(y(u))]^2\, du \\ &\quad - \tfrac{1}{2} \int_0^t [g(y(u))]^2 a'(t-u)\, du, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{d}{dt} E(y(t)) &= -g(y(t))\, p'(t) - \frac{1}{2} a(t)[g(y(t))]^2 + \frac{1}{2} \int_0^t a'(t-u)[g(y(t)) \\ &\quad - g(y(u))]^2\, du \\ &\leqslant -g(y(t))\, p'(t), \qquad t \geqslant 0. \end{aligned} \tag{29}$$

In this calculation we have assumed $a(0)$ finite, but all the steps can be justified if $a(t)$ is integrable at 0 (see [23, p. 357]). As long as $y(t)$ remains in I, we may use (27) and (28) to obtain

$$\begin{aligned} \frac{d}{dt} E(y(t)) &\leqslant \beta \mid y(t) - x_\infty \mid \mid p'(t) \mid \leqslant K \left| \frac{\alpha}{2} (y - x_\infty)^2 \right|^{1/2} \mid p'(t) \mid \\ &\leqslant K \mid G(y(t)) \mid^{1/2} \mid p'(t) \mid \leqslant K \mid E(y(t)) \mid^{1/2} \mid p'(t) \mid \end{aligned}$$

for a suitable constant $K > 0$. By the standard comparison theorem for differential inequalities [9, p. 29], we see that as long as $y(t) \in I$, $E(y(t))$ is no greater than the maximal solution of

$$r' = Kr^{1/2} \,|\, p'(t)|, \qquad r(0) = E(y(0)),$$

or

$$E(y(t)) \leqslant \left[\{E(y(0))\}^{1/2} + \frac{K}{2} \int_0^t |\, p'(v)|\, dv \right]^2.$$

For any given $E(y(0))$ with $y(0) = c \in I$, we can choose $\int_0^\infty |\, p'(v)|\, dv$ small enough that $E(y(t))$ remains small enough to imply $y(t) \in I$ for all $t \geqslant 0$. This in turn implies the boundedness of $y(t)$. This completes the proof of Theorem 7.

The argument used in the proof of Theorem 7 also gives some information about the unperturbed equation (24). Here, (29) gives $(d/dt)\, E(x(t)) \leqslant 0$ for (24), from which we can deduce that $x(t)$ is monotone, and hence, using Theorem 5, that $x(t)$ tends monotonically to x_∞ as $t \to \infty$. This contains the principal results of [24] and [31].

COROLLARY. *Suppose the hypotheses H_a' and H_g' are satisfied and that there exists x_∞ with $g(x_\infty) = 0$, $g'(x_\infty) < 0$. Then if c is chosen so that $g'(x)$ is strictly decreasing on $c \leqslant x \leqslant x_\infty$ (if $c < x_\infty$) or $x_\infty \leqslant x \leqslant c$ (if $c > x_\infty$), the solution $x(t)$ of (24) tends monotonically to x_∞ as $t \to \infty$.*

Theorem 7 is an analog for the hypotheses H_p', H_a', and H_g' of Theorem 2, Corollary 2 for the hypotheses H_f, H_a, H_g. It is natural to conjecture that under the hypotheses H_p', H_a', H_g' some analog of Theorem 4 dealing with the effect of a perturbation of g should be valid. However, because of the difficulty in studying the variational equation in this case, no such result is obvious.

7. The perturbation results Theorem 2, Corollary 2 and Theorem 7 in the respective cases $\int_0^\infty a(v)\, dv < \infty$ and $\int_0^\infty a(v)\, dv = \infty$ can be combined formally. We rewrite (3) as

$$\frac{x_\infty}{\int_0^\infty a(v)\, dv} = \frac{f(\infty)}{\int_0^\infty a(v)\, dv} + g(x_\infty) \tag{30}$$

and interpret $1/\int_0^\infty a(v)\, dv = 0$ if $\int_0^\infty a(v)\, dv = \infty$. Then (30) contains both (3) and (26), and

$$g'(x_\infty) < 1 \Big/ \int_0^\infty a(v)\, dv \tag{31}$$

contains both (18) and $g(x_\infty) < 0$ (if $\int_0^\infty a(v)\,dv = \infty$). This suggests that we may define an equilibrium value of (1) as a solution x_∞ of (30). Then Theorem 2, Corollary 2 and Theorem 7 can be interpreted as saying that condition (31) implies a kind of stability for the equilibrium value x_∞. This stability is not the same as for ordinary differential equations, since a nonconstant solution of (1) may tend to an equilibrium value x_∞ for which

$$g'(x_\infty) > 1\Big/\int_0^\infty a(v)\,dv,$$

as has been mentioned earlier. However, it does have similar properties for perturbations of both f and g to asymptotic stability for ordinary differential equations. A first-order autonomous ordinary differential equation is, of course, the special case of (24) with $a(v) \equiv 1$.

It would be reasonable to attempt to model population of interacting species with growth rates depending on the population sizes and a probability of death for each species depending on age by a system of nonlinear renewal equations of the form

$$x_i(t) = f_i(t) + \int_0^t g(x_1(t-v), x_2(t-v),\ldots, x_n(t-v))\, a_i(v)\,dv \qquad [i = 1, 2, .., n].$$

It is not yet known whether the results of [21] or [23] for a single equation can be extended to such systems. If so, it would be plausible to conjecture that the results of the present paper also can be generalized to systems and thus that the classification of critical points for systems of ordinary differential equations can be fitted into the same framework.

Another question worthy of attention is the one solved in a special case in [20]. There it was shown that a periodic forcing function produced an asymptotically periodic solution. The methods of the present paper exclude the possibility of periodic forcing functions, but the question of the effect of such forcing functions is of interest.

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